LC HL – Differential Calculus – Booklet

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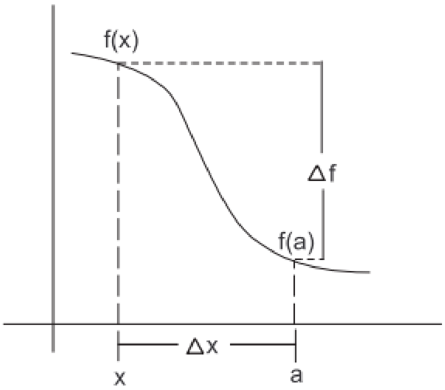
Differentiation is the process of calculating a derivative. The derivative of a function represents an infinitesimal change in the function with respect to whatever parameters it may have. For example, if the independent variable x of a function $f(x)$ is increased by a small amount Δx ("delta x") it will cause a corresponding change Δf of $f(x)$. The ratio $\frac{\Delta f}{\Delta x}$ is a measure of the *rate of change* of f with respect to x. The limit value, $\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$ when Δx tends to zero (if it exists) is the first derivative of $f(x)$ with respect to x which describes the instantaneous change of f at a given point x.



Another way of thinking of differentiation is in terms of slopes or gradients. For all curves other than linear (straight-line) curves, the gradient of the curve can change at each point along the axis. This means that it is extremely difficult to calculate the gradient of the curve at any given point. Differentiation can be used to find the gradient of any curve by calculating the slope of the tangent to the curve at the point in question. Therefore differentiation can be thought of as calculating the slope of a curve at a given point.

Rates of Change

Differentiation can also be defined in terms of rates of change, but what exactly do we mean when we say rates of change ? Consider the following example. Imagine you are driving from Limerick to Cork. You start your journey at midday and obey all the speed limits. Assume that when you reach your destination in Cork you have travelled exactly 100 kilometers and that it took you exactly 2 hours. It is easy to see that we have averaged 50 kilometers per hour during this journey. This means that your **average speed** was 50 km/h. If you had looked at your speedometer at a particular time during your journey you would have seen the **instantaneous speed** you were travelling.

The average speed of the car during the journey is measured by dividing the distance it has travelled by the time it has taken to travel that distance. In this example

$$\frac{\text{Distance travelled}}{\text{Time taken}} = \frac{100}{2} = 50 \text{ kilometers per hour}$$

Average speed is the rate of change of distance with respect to time and is calculated from the ratio of distance travelled to the time taken. Instantaneous speed, as opposed to average speed, is the rate of change of distance with respect to time *at a specific time* and because of this we cannot calculate instantaneous speed from a ratio because the denominator - at a specific time - is zero. To overcome this problem we calculate the derivative. The derivative of a function is a measure of the instantaneous rate of change of the function - that is, it is a way of measuring how the function changes at each separate point.

1. First Principles

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Procedure

1. $f(x+h)$
2. $f(x+h) - f(x)$
3. $\frac{f(x+h)-f(x)}{h}$
4. $\log_{h-0} \frac{f(x+h)-f(x)}{h}$

Eg. 1

Differentiate with respect to x from first principles: $2 + 3x - 2x^2$

Text & Tests 7: Pg.64 Q's: 1(ii), 2(iii), 5(ii)

Past Exams: 2014 Q. 4(a)

Eg. 2

Use first principles to find the derivative of $f(x) = 2x^2 - 3x - 2$.

(i) Use this to find the slope of the tangent to the curve at the pint (3,7)

(ii) Find the equation of the tangent to the curve at the point (3,7)

Text & Tests 7: Pg.64 Q's: 3, 6, 7

1. Rules

a. Basics

The Rule:

If $f(x) = x^n$ then $f'(x) = nx^{n-1}$

Eg. 1

Find the derivative of the following

$$y = 2x^5 + x^3$$

$$y = \frac{1}{x^4}$$

Eg. 2

Find the derivative of the following

$$8 + x^2 - \frac{1}{x}$$

Product Rule:

$$y = uv$$

$$\Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = (first)(derivative\ of\ second) + \ (second)(derivative\ of\ first)$$

Eg. 1

A question such as $y = (2x + 5)(x^2 - 3)$ we can multiply out the brackets and then differentiate.

Alternatively we can use The Product Rule, which is given on The Formulae & Tables Booklet:

$$y = \overset{u}{(2x + 5)}\overset{v}{(x^2 - 3)}$$

Eg. 2

Find the derivative of the following: $y = \sqrt{x}(x + 2)$

The Quotient Rule

A question with a fraction such as $y = \frac{3x-2}{x^2+3}$ we can't just differentiate the top and bottom.

We must use the Quotient Rule, which is in The Formulae & Tables Booklet:

$$y = \frac{u}{v}$$

$$\Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(bottom)(dirvative\ of\ top) - (top)(derivative\ of\ bottom)}{(bottom)^2}$$

Eg. 1

$$y = \frac{3x-2}{x^2+3}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Eg. 2

Differentiate the following with respect to x,

$$y = \frac{1}{2+5x}$$

The Chain Rule

When given a question with a power such as $y = (4x^2 - 2)^7$ we could multiply out the brackets 7 times and then differentiate but this would be difficult and a waste time. Therefore we use **The Chain Rule**.

A formula for The Chain Rule is given in The Formulae & Tables Booklet but it is confusing so we will use a far easier rule:

$$\frac{dy}{dx} = (power)(bracket)^{power-1}(differentiation\ of\ bracket)$$

Lets illustrate this with the example from above:

$$y = (4x^2 - 2)^7$$

Eg.2

Differentiate $(3x^3 - 2x^2 + 2)^4$ with respect to x .

Eg. 3

Given that $y = (x^2 - 2x - 3)^3$, show that $\frac{dy}{dx} = 0$ when $x = 1$.

b. Trigonometry

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}\tan x = \sec^2 x$$

Eg. 1

Differentiate with respect to x the following:

$$y = \cos(7x - 3)$$

Eg. 2

$$\cos^4 x$$

Eg. 3

$$y = \frac{1 + sinx}{cosx}$$

Eg. 4

$$y = 2x - sin2x$$

c. Logs

You should use your laws of logs to bring log problems to their simplest, natural log form (y = In(x)).

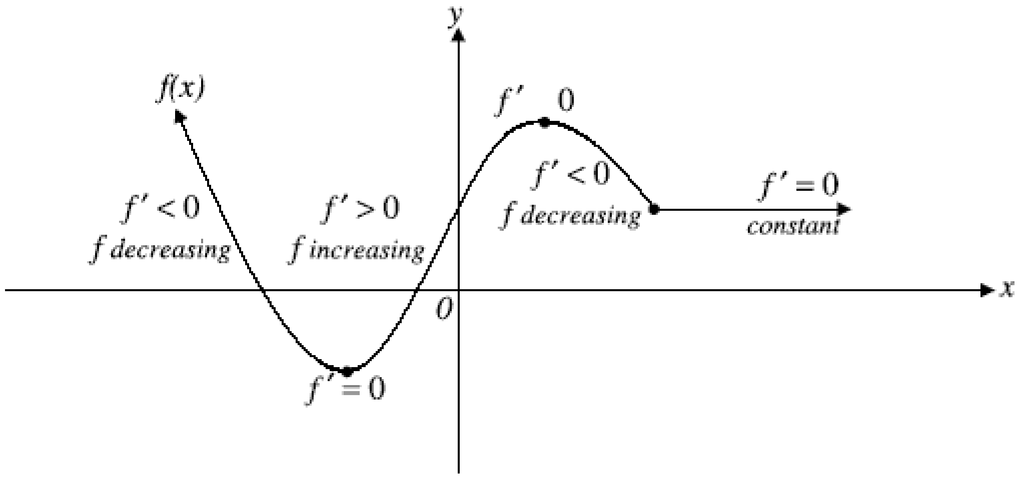
d. Exponentials

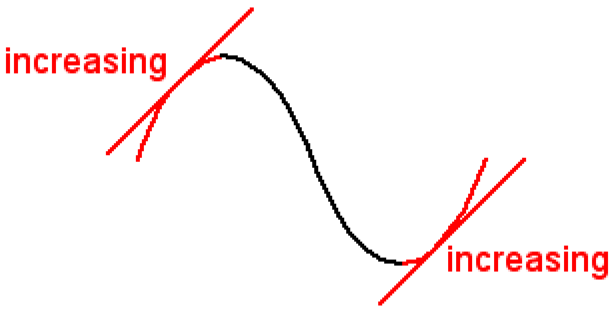
e. Inverse

2. Tangents & Increasing or Decreasing

General Rules of Increasing or Decreasing Curves

$$\frac{dy}{dx} > 0 \ldots\ldots Increasing$$
$$\frac{dy}{dx} < 0 \ldots\ldots Decreasing$$





f '(x)	+	-
f(x)	f increasing	f decreasing

Finding the range of values for when a Function is Increasing or Decreasing:

1. Find the first derivative
2. Find the second derivative
3. Leave second derivative equal to zero, find value(s) for x. (This is called the point of inflection, we'll look at this later)
4. Test any value for x either side of the POI

Eg. 1

Find the range of values for which the following function is increasing or decreasing

$$f(x) = x^3 - 3x^2 - 9x + 9$$

1. Find $\frac{dy}{dx}$

2. Find $\frac{dy}{dx}$

3. Solve for x , $\frac{dy}{dx} = 0$

4. Test any value less/greater than the value you found above.

5. Write the range

Eg. 2

Find the range of values for which the following function is increasing or decreasing

$$y = 2x^3 - 15x^2 + 36x$$

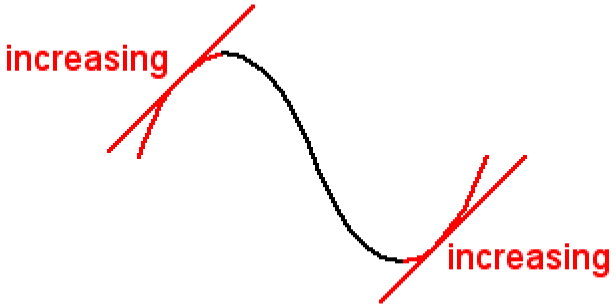
Eg. 3

Find the range of values for which the following function is increasing or decreasing

$$y = \sqrt{x + 2}$$

Finding the Equation of the tangent to a curve

1. Find $\frac{dy}{dx}$
2. Evaluate $\frac{dy}{dx}$ for the given value of x. (This is the Slope)
3. Use the point given and $(y - y_1) = m(x - x_1)$ to find the equation of the tangent.



Eg. 1

Find the equation of the tangent to the curve at (-1, -2).

$$y = \frac{x-1}{x+2}$$

Eg. 2

Find two points on the curve where the tangents to the curve are parallel to the line.

Curve: $f(x) = x^3 - 3x^2 - 5x + 10$

Line: $f(x) = 4x - 7$

Text & Tests 7

Basics: Pg.97/98 Q's: 2, 4,5, 6, 10, 11, 14, 17, 18(ii), 20, 22.

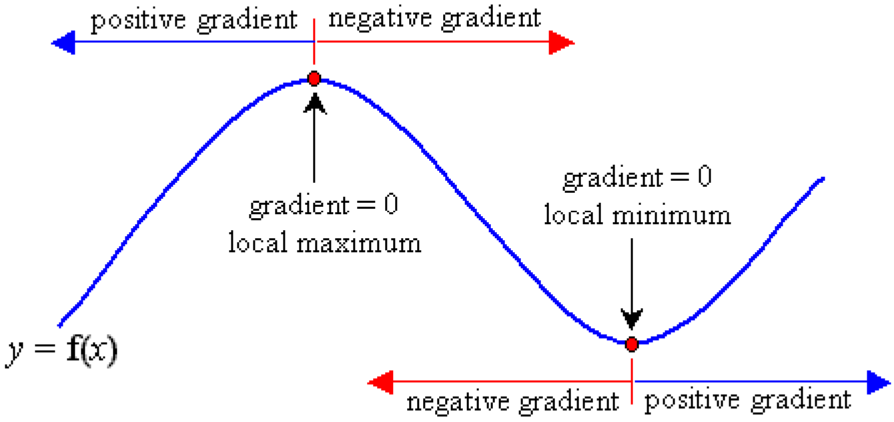
Advanced: Pg. 98 Q's: 23 & 24.

Exam Papers: 2012 Q6(b), 2014 Q4 (b)

4. Max/Min & Points of Inflection

The gradient of the curve depends upon where we are on the curve. The slope is defined as the slope of the tangent 'drawn' to the curve at a given point P.

To do this we find the slope.



To Find the Stationary Points

Procedure

1. Find $\frac{dy}{dx}$
2. Let $\frac{dy}{dx} = 0$
3. Find the values for x, then find a corresponding value for y by using the original function

Eg. 1

Find the stationary points of the following function

$$y = x^3 - 2x^3 - 4x$$

Eg. 2

Find the stationary points of the following function

$$y = e^x - x$$

Eg. 3

Find the stationary points of the following function:

$$y = x^3 - 9x^2 + 24x - 20$$

Determining a Stationary Point is The Local Maximum or Minimum

$$At\ a\ local\ max\ \ldots.\frac{dy}{dx} = 0\ \ldots.and\ \ldots.\frac{d^2y}{dx^2} < 0$$

$$At\ a\ local\ min\ \ldots.\frac{dy}{dx} = 0\ \ldots.and\ \ldots.\frac{d^2y}{dx^2} > 0$$

Procedure:

1. Find $\frac{dy}{dx}$
2. Let $\frac{dy}{dx} = 0$
3. Find the values for x.
4. Find $\frac{d^2y}{dx^2}$
5. Using values for x from Step 3, perform the second derivative test (above).

Eg. 1

Find the stationary points of the following function and determine if the points are the local max/min

$$y = x^3 - 2x^3 - 4x$$

Text & Tests 7
Basics: Pg.103. Q's: 2, 3, 5(ii), 8, 9, 13, 14, 16, 19.
Exam Papers: 2012 Q9 (a)

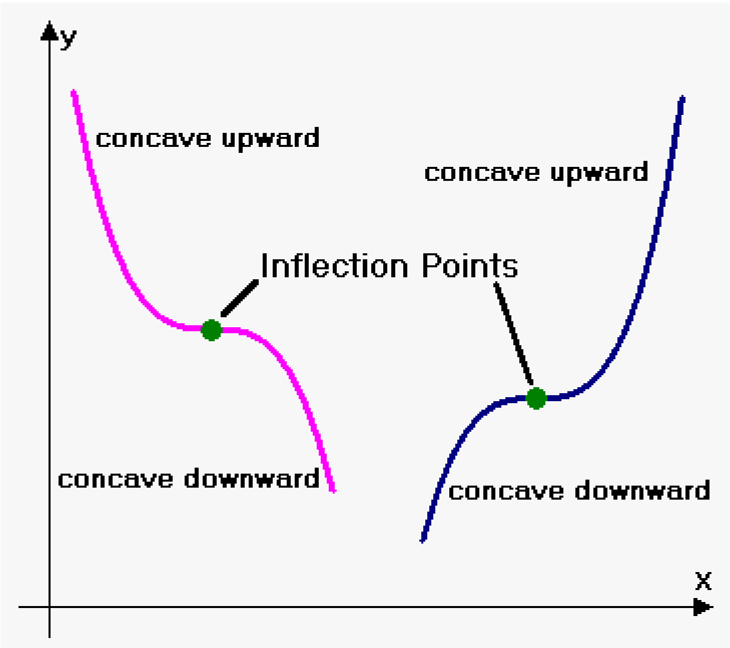
Eg. 3

Find the stationary points of the following function and determine if the points are the local max/min

$$y = x^3 - 9x^2 + 24x - 20$$

Finding the point of inflection of a curve

In differential **calculus**, an **inflection point**, **point of inflection**, flex, or **inflection** (**inflexion**) is a **point** on a curve at which the curve changes from being concave (concave downward) to convex (concave upward), or vice versa.



Procedure

1. Find $\frac{d^2y}{dx^2}$
2. Solve for x, $\frac{d^2y}{dx^2} = 0$
3. Find the corresponding y value, using the original function

Eg. 1

Find the point of inflection of the curve given

$$y = x^3 - 3x^2 - 2$$

5. Rational Functions

Text & Tests 7

Basics: Pg.104. Q's: 12, 18… No need to use geogebra

6. Rates of Change

Any given function is simply just an expression of how y changes as x progresses.

A typical application of differentiation is to calculate the rate of change of some given variable, such as displacement, speed, acceleration, mass or volume over time.

A function *'s'* might represent the displacement (distance) an object travel as time *'t'* progresses.

$$s = distance\ in\ terms\ of\ time(t)$$

$$\frac{ds}{dt} \ldots the\ rate\ of\ change\ of\ s\ with\ respect\ to\ time \ldots \boldsymbol{SPEED}!!$$

$$\frac{d^2s}{dt^2} \ldots the\ rate\ of\ change\ of\ speed\ with\ respect\ to\ time \ldots \boldsymbol{ACCELERATION}!!$$

Eg. 1

A particle is moving in a straight line its distance, s meters, from a fixed point o after t seconds is given by:

$$s = t^3 - 9t^2 + 15t + 2$$

(i) Find its velocity for any time t
(ii) Find the exact velocity at 6 seconds
(iii) The distance from o when the particle is at rest
(iv) Its acceleration after 4 seconds

Text & Tests 7
Basics: Pg.120. Q's: 7,8,9,10,11,13
Exam Papers: 2014 Sample Q9 (b)

Related Rates of Change

Finding the rates of change for two or more variables

If given the function for the area of a circle $A = \pi r^2$ we could easily find the rate of change of the area of a circle with respect to r, the radius $\frac{dA}{dr}$. But in most practical examples people are only interested in the rate of change with respect to time t, $\frac{dA}{dt}$.

$\frac{dA}{dt}$ is the equivalent to saying how much will the area change over a given time t, of course this depends of how much the radius changes over time.

$\frac{dA}{dt} = \frac{dA}{dr}.\frac{dr}{dt}$, because the area is dependent on how much the radius changes, and the radius is dependent on how much it will change over time.

Procedure:

1. Write down the rate you require
2. Write down the rate you're given
3. Link these rates using $\frac{dy}{dx} = \frac{dy}{d?}.\frac{d?}{dx}$
4. Find an equation that links these rates.

Eg. 1

A stone is thrown into a pond, causing circular ripples to move away from the point of impact at a rate of 10cm per second. When a circular ripple has a radius of 50cm find the following:

(i) Find the rate of change of the circles circumference
(ii) Find the rate of change of the circle area

Eg. 2

A company that grows beans wishes to pack their product in closed catering pack cylindrical tins. Each tin must have a volume of 332.75π cm^3 and the minimum possible surface area. Find the dimensions of the tin.

Eg. 3

Water is poured into a cone so that the volume of water is increasing at $3cm^3$ per second. The cone has a perpendicular height h, a radius r and an angle at its peak of 60^o.

(i) Represent this in a diagram
(ii) Find the volume V, of the cone in terms of the radius r.
(iii) Find the rate of increase of the circular surface area of the water when the radius is, r=2cm.

Eg. 3

An open cylindrical tank of water has a hole near the bottom. The radius of the tank is 52cm. The hole is a circle of radius 1cm. The water gradually drops as the water escapes through the hole.

Over a certain period of 20mins the height of the surface of the water is given by the formula:

$$h = \left(10 - \frac{t}{200}\right)^2$$

(i) Draw a representation of the water tank
(ii) What is the height of the water at t=0
(iii) How many seconds will have elapsed when the water is at 64cm
(iv) Find the rate at which the volume of the water tank is decreasing when the height is 64cm
    a. Why is the rate of change negative

Text & Tests 7
Basics: Pg.123. Q's: 4,5,7,8,11,12,14,
Exam Papers: 2014 Sample Q9 (c)
Applying Max/Min
Pg 114. Q's: 2,4,5,6,7,8,9,11,13,14
Exam Papers: 2014 Q7, 2012 Q8

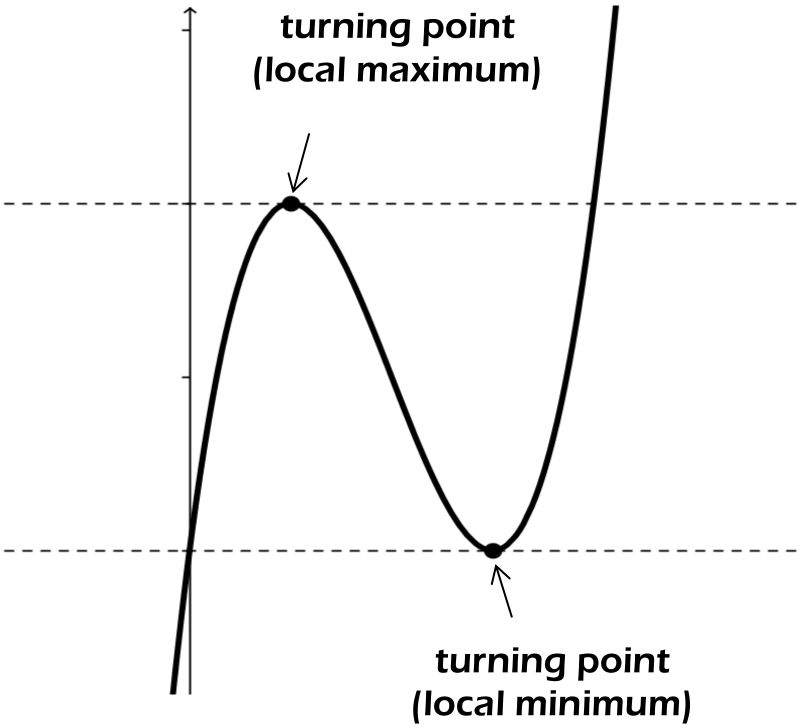
7. Interpreting Graphs

- Using differentiation to find turning points and sketch graphs

Differentiation can also be used to find the turning point(s) of the graphs of any function.

The derivative $\frac{dy}{dx}$ is basically the slope of the tangent at any point on a curve.

At a turning point (maximum or minimum point), the tangent is totally flat!



If the tangent is flat, that means its slope = 0…so at a turning point, $\frac{dy}{dx} = 0$!

To find the turning point(s) of a graph…

find the derivative $\frac{dy}{dx}$,

put this derivative = 0 and solve the equation you end up with

…this will give you the x co-ordinate(s) of the turning point(s),

put the x value(s) back into the **original function** to find the matching y value(s), if necessary decide which is a local maximum and which is a local minimum

…look at the y values…the larger one is higher up and this is the maximum.

If the Examiner asks you to sketch a curve, you just need to find the turning point(s) as described above. It is a good idea to use your calculator to find a couple of points around the turning point(s) to help get a feel for the shape.

You then just plot these points and join them together with a smooth curve.

8. Sample Questions

Turning Points

Question 1

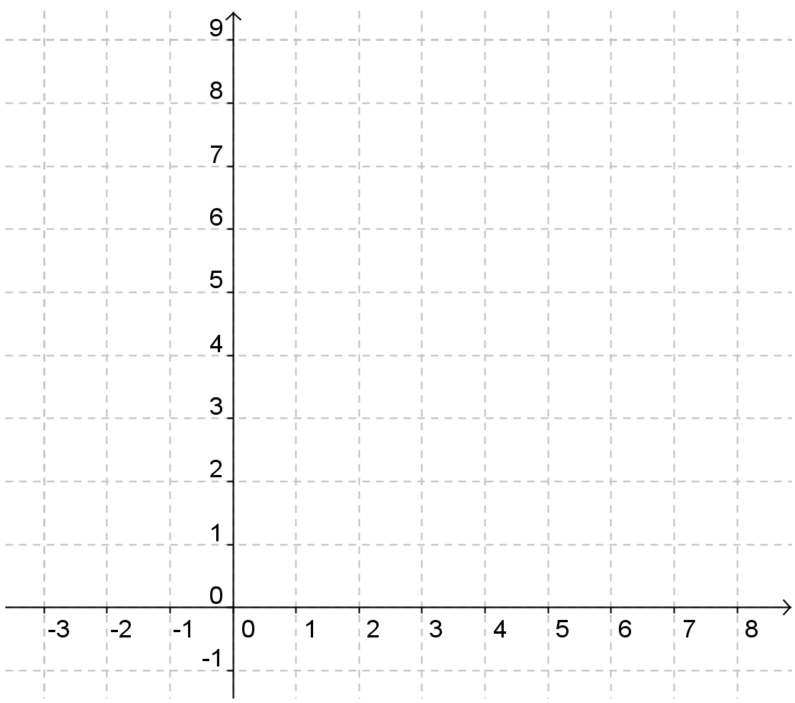
A function is defined as $f(x) = x^3 - 6x^2 + 9x + 4$.

(i) Find the co-ordinates of the turning points of the graph of this function.

(ii) Which of these points is a local maximum?

(iii) Which of these points is a local minimum?

(iv) Draw a rough sketch of the curve of this function.



Question 2

(i) Find the turning points of the function $y = 27x - x^3$.

(ii) Determine the nature of each of these turning points.

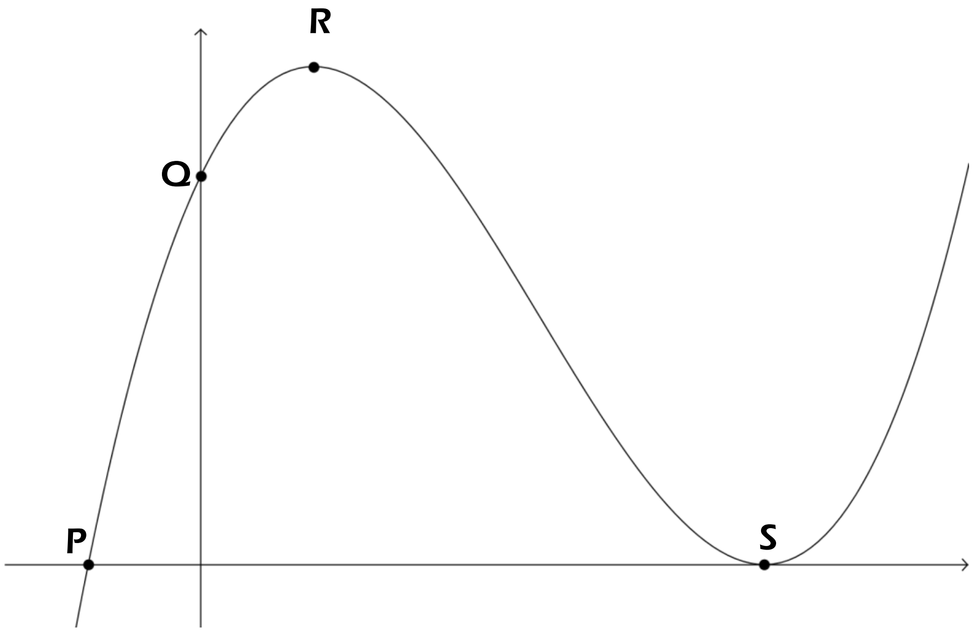
Question 3

(i) Find the turning points of the function $y = x^3 - 6x^2$.

(ii) Determine the nature of each of these turning points.

Question 4

The graph below shows the function $f(x) = x^3 - 9x^2 + 15x + 25$.



(i) Show that $x = -1$ is one root of the equation $x^3 - 9x^2 + 15x + 25 = 0$.
   a. Hence write down the co-ordinates of the point P.

(ii) The curve intersects the y-axis at the point Q.
   a. Find the co-ordinates of Q.

(iii) Will the slope of the tangent at the point Q be positive or negative?
   a. What does this indicate about the curve at this point?

(iv) R and S are the turning points of the curve.
   a. Find the co-ordinates of R and S.

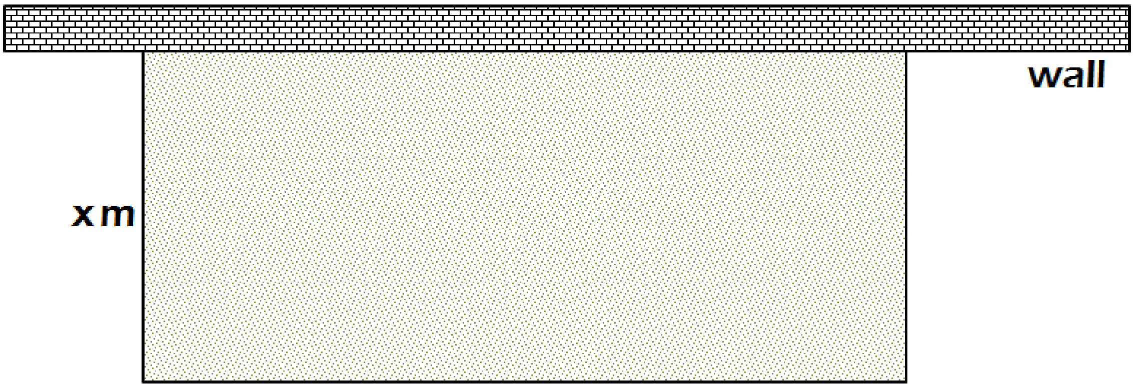
Question 5

A rectangle has a perimeter of 40 cm.
Let x be the length of the rectangle.

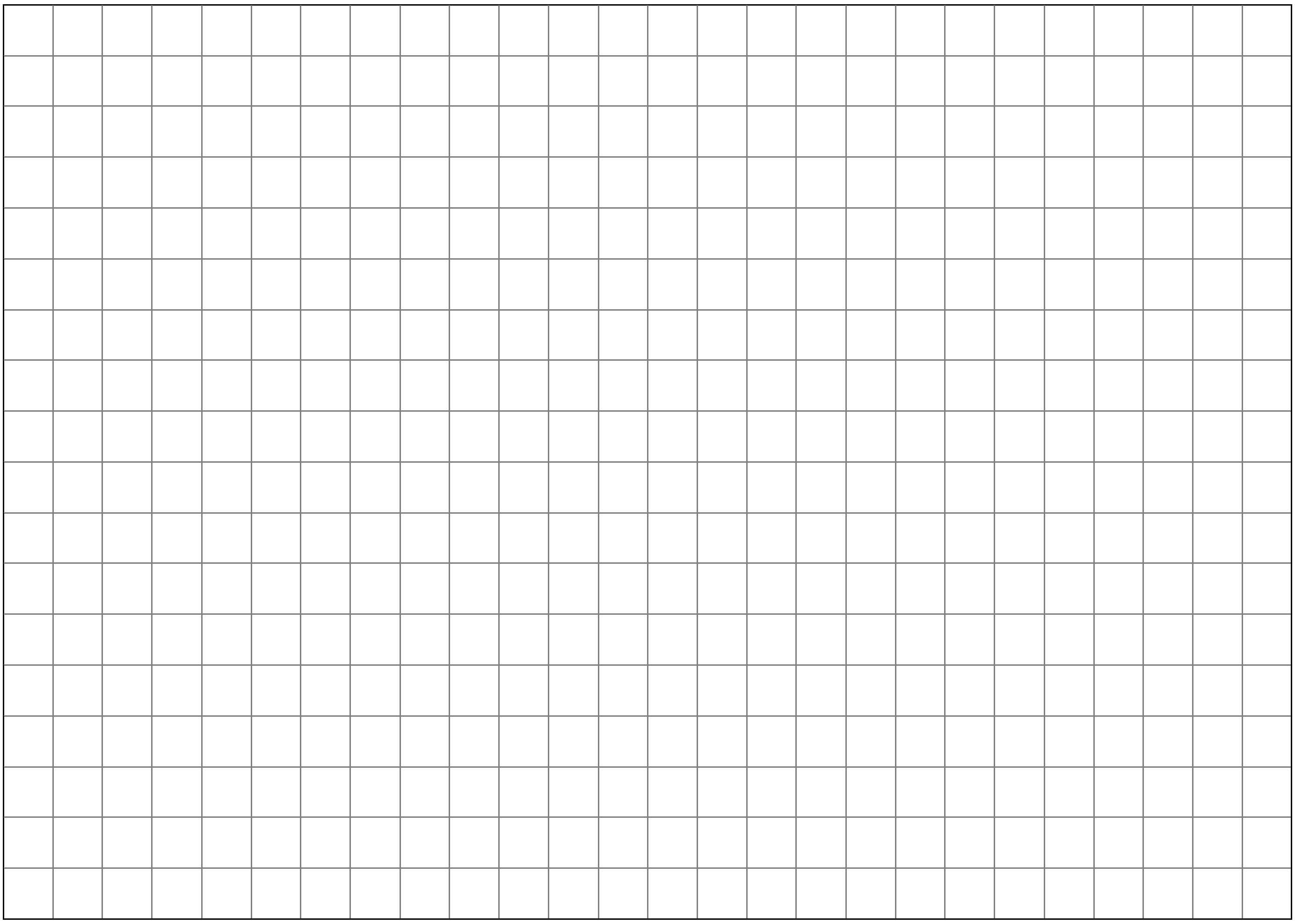
(i) Write the width of the rectangle in terms of x.
(ii) Show that the area of the rectangle (A) can be given by $A = 20x - x^2$.
(iii) Find the maximum possible area of the rectangle.

Question 6

A farmer intends to fence off a rectangular paddock in one of his fields.
He intends to use the outside wall as one of the sides of this paddock.
He has 60 m of fencing with which to create the other three sides.
Let x be the width of the paddock as shown below.



(i) Write the length of the paddock in terms of x.
(ii) Write an expression in x to represent the area of the paddock.
(iii) Find the maximum possible area of the paddock.



Question 7

A landlord owns a block of 48 apartments.

With maintenance and other costs, he realises that renting all 48 apartments might not maximise his profits.

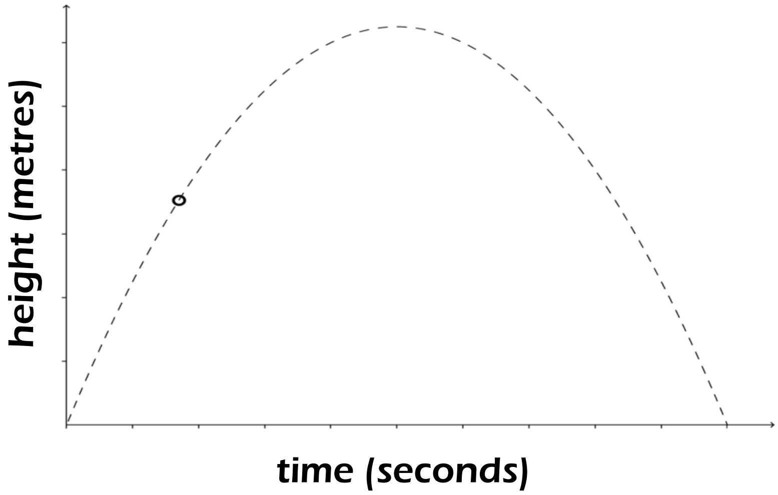
The profit (P) that he makes on renting n apartments is given by the formula $P = 4342 + 532n - 7n^2$.

(i) Find the number of apartments he should rent to maximise his profits.
(ii) Find his maximum possible profit.
(iii) By how much are these profits decreased if he rents all 48 apartments?

Question 8

A model rocket is fired into the air.
The height of the rocket (H) in metres after t seconds is given by the formula

$H = 25t - \frac{1}{2}t^2$



(i) Find the height of the rocket after 4 seconds.

(ii) After how many seconds does the rocket reach its maximum height?

(iii) Find the maximum height reached by the particle.

(iv) After how many seconds will the rocket land back on the ground?

Question 9

As part of an ecological study, a group of biologists measured the population of two animal species in a particular area over a period of 12 months.

The population (P) of each species after t months is given by the formulas below.

Species 1: $P = -t^2 + 18t + 14$

Species 2: $P = t^2 - 14t + 110$

(i) Find the initial population of each species.

(ii) After how many months were the populations of the two species equal?

(iii) For how many months was the population of species 1 higher than the population of species 2?

(iv) Find the minimum measured population of species 2.

(v) Find the maximum measured population of species 1.

(vi) Theoretically, how long after the group finished studying the animals would the population of species 1 drop to zero?

Extra Sample Questions

Question 10

A function is defined as follows: $y = x^3 + px^2 + qx - 52$.

(i) The slope of the tangent to this curve at the point $(-3, -34)$ is 21.

Find the values of p and q.

(ii) K is the point where the curve cuts the y-axis.

Find the equation of the tangent to this curve at the point K.

(iii) Find the two points at which the slope of the tangent to the curve is -15.

(iv) Find the turning points of the curve.

Determine the nature of each of these turning points.

(v) Find the point of inflection of the curve.

(vi) Draw a rough sketch of the curve.

Question 11

(a) A particle travels in a straight line so that the distance in metres (S) travelled after t seconds is given by the formula $S = t^3 - 4t^2 + 4t$.

(i) Find the speed of the particle after 3 seconds. [5]

(ii) Find the acceleration of the particle after 1 second. [5]

(iii) Find the times at which the particle is momentarily at rest. [5]

(b) A cylinder has radius r cm and height 4r cm.

The radius of the cylinder is increasing at a rate of 0.5 cm/sec.

Find the rate at which the volume is increasing when the radius is 6 cm. [10]

Question 12

(a) A particle P moves in a straight line from a fixed starting point O.

After t seconds the displacement in metres (S) from O is given by

$$S = t^3 - 2t^2 + 4t$$

(i) Find the distance from P to O after 2 seconds. [5]

(ii) Find the times at which the velocity of P is 4 m/sec. [5]

(b) A circle has radius r cm, circumference C cm and area A cm.

(i) Show that $\frac{dC}{dA} = \frac{1}{r}$. [10]

(ii) The area of a circle is increasing at the a rate of 2 cm^2/sec.

Find the rate of increase of the circumference when r = 3 cm. [5]

Question 13

(a) An object is projected upwards and its height above the point of projection in metres (H) after t seconds is given by $H = 600t - 5t^2$.

(i) After how many seconds is the object momentarily at rest? [5]

(ii) Find the greatest height in kilometres the object reaches. [5]

Question 14

(b) A curve is defined as $\frac{x+2}{2x-3}$.

(i) Show that this curve has no turning points. [10]

(ii) Explain why this curve is always decreasing. [5]

Question 14

(a) A curve is defined as $y = x^3 - 3x^2 - 2$.

(i) Find the turning points of this curve. [10]

Determine the nature of each of these turning points. [10]

Question 15

A function is defined as follows: $y = x^3 + px^2 + qx - 52$.

(i) The slope of the tangent to this curve at the point $(-3, -34)$ is 21.
Find the values of p and q.

(ii) K is the point where the curve cuts the y-axis.
Find the equation of the tangent to this curve at the point K.

(iii) Find the two points at which the slope of the tangent to the curve is -15.

(iv) Find the turning points of the curve.
Determine the nature of each of these turning points.

(v) Find the point of inflection of the curve.

(vi) Draw a rough sketch of the curve.

Question 16

Find $\frac{dy}{dx}$ for each of the following.

(i) $y = 2x^4 + 3x^3 - 2x^2 + x - 3$

(ii) $y = 9x^{-1} - 4x^{-3}$

(iii) $y = \frac{2}{3}x^3 + \frac{1}{4}x^{-2}$

Question 17

Find f '(x) for each of the following.

$f(x) = 6x^6 - x^3 + 7x - 29$

$f(x) = 5x^{-2} - x^{-3}$

$f(x) = \frac{5x^6}{3} + \frac{2x^5}{5}$

Question 18

A landlord owns a block of 250 apartments.

With maintenance and other costs, he realises that renting all 250 apartments might not maximise his profits.

The profit (P) that he makes on renting x apartments is given by the formula $P(x) = 3200 + 2520x - 7x^2$.

(i) Find the number of apartments he should rent to maximise his profits.

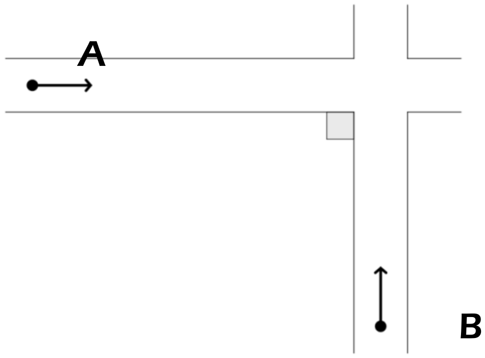
(ii) Find his maximum possible profit.

Question 18

Two cars, A and B, are heading towards a right-angled intersection as shown in the diagram opposite.

A is currently 600m from the intersection travelling at a speed of 20m/sec.

B is currently 400m from the intersection travelling at a speed of 10m/sec.



(i) Show that the distance (D) between the two cars at any time (t) is given by the following formula.

$$D(t) = 10\sqrt{5200 - 340t + 5t^2}$$

(ii) What is the shortest distance there will ever be between the two cars?